

Fractional Supersymmetry and Quantum Mechanics

STÉPHANE DURAND*

*Department of Physics
McGill University
3600 University Street
Montréal, PQ, H3A 2T8, Canada*

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Abstract

We present a set of quantum-mechanical Hamiltonians which can be written as the F^{th} power of a conserved charge: $H = Q^F$ with $[H, Q] = 0$ and $F = 2, 3, \dots$. This new construction, which we call *fractional* supersymmetric quantum mechanics, is realized in terms of paragrassmann variables satisfying $\theta^F = 0$. Furthermore, in a pseudo-classical context, we describe *fractional* supersymmetry transformations as the F^{th} roots of time translations, and provide an action invariant under such transformations.

* E-mail address: durand@hep.physics.mcgill.ca

1. Introduction

Supersymmetric (SUSY) quantum mechanics has applications in nuclear and atomic physics, and is also a simpler framework to understand new ideas before extending them into SUSY field theories. In SUSY quantum mechanics, the Hamiltonian H is written as the square of a conserved supercharge: $Q^2 = H$. In its simplest realization, it describes a spin-1/2 particle. Recently, the para-supersymmetric (PSUSY) generalization of SUSY quantum mechanics was developed^[1,2]. A PSUSY quantum-mechanical Hamiltonian of order 3, which describes a spin-1 particle, is realized as^[1]: $Q^3 = QH$ with $Q^2 \neq H$ (where Q is now a conserved para-supercharge). In general, a PSUSY quantum-mechanical Hamiltonian of order M is of the form $Q^M = Q^{M-2}H + \dots$ and describes a spin- $(M-1)/2$ particle^[2].

In this letter, we present an alternative generalization of SUSY quantum mechanics which we call *fractional* supersymmetric (FSUSY) quantum mechanics of order F . Hamiltonians of this new kind are expressed as the F^{th} power of a conserved *fractional* supercharge: $Q^F = H$ with $F = 2, 3, \dots$. This means that the FSUSY transformations are the F^{th} roots of time translation. We discuss these transformations in a *pseudo-classical* Lagrangian context, and provide a FSUSY invariant action (a fractional-superspace formulation is given in Ref. [3]).

The internal space of the quantum-mechanical systems is described by a paragrassmann variable θ of order F satisfying $\theta^F = 0$. In matrix form, the Hamiltonians are realized in terms of $F \times F$ matrices. The spectrum of the one-dimensional harmonic oscillator (or equivalently the two-dimensional constant magnetic field) is found to be F -fold degenerate above the ground state, which can be both *unique* and *degenerate*. We also describe a conformal symmetry, namely dilations, as a dynamical symmetry for the $1/x^2$ potential. In Ref. [4], an alternative realization of the FSUSY algebra is given in which q -deformed relations are found among different conserved charges.

2. Paragrassmann variables

In this section, we introduce generalized variables which interpolate between ordinary bosonic and fermionic ones. They can be interpreted either as general-

ized coordinates, or generalized creation and annihilation operators. The latter interpretation is more relevant in the present quantum-mechanical context, but the former point of view is used in Ref. [3] when discussing the fractional superspace formulation of FSUSY transformations. The notation, however, will be more reminiscent of the generalized coordinates interpretation. A more complete presentation of this formalism is given in Ref. [4].

We introduce a paragrassmann variable θ of order F , and its derivative $\partial \equiv \partial/\partial\theta$, which satisfy

$$\begin{aligned} \theta^F &= 0, & \partial^F &= 0, & F &= 1, 2, \dots \\ (\theta^{F-1} &\neq 0, & \partial^{F-1} &\neq 0). \end{aligned} \tag{1}$$

In order to be able to recover the 3 different limits which we describe below (fermionic, bosonic and “null”), we take the generalized commutation relation between θ and ∂ to be

$$[\partial, \theta]_q \equiv \partial\theta - q\theta\partial = \alpha(1 - q), \tag{2}$$

where α is a free parameter and $q \in \mathcal{C}$ a *primitive* F^{th} root of unity:

$$q^F = 1 \quad (q^n \neq 1 \text{ for } 0 < n < F). \tag{3}$$

By a *primitive* root, we mean a root satisfying the condition in parentheses; for instance, $q \neq \pm 1$ for $F = 4$. [We will see below that the condition (3) is actually a consequence of (1) and (2).] First, note that the null limit $F = 1$ ($q = 1$), that is $\theta = \partial = 0$, is well-defined since the r.h.s of (2) is zero for $q = 1$ (and for finite α). In previous works^[5,6] on paragrassmann variables, the r.h.s of (2) was chosen to be 1 instead of $\alpha(1 - q)$, which is inconsistent in the limit $F = 1$ (which we use in Ref. [3,4]). Second, note that we recover the ordinary grassmann case ($q = -1$) for $F = 2$, *i.e.*, $\{\partial, \theta\} = 2\alpha$. Third, for some choices of α , we also recover (within factors) the bosonic case ($q = 1$) for $F \rightarrow \infty$. For instance, for $\alpha = F$ we find that the r.h.s of (2) is finite and non-zero:

$$\lim_{F \rightarrow \infty} F(1 - q) = -2\pi i. \tag{4}$$

Strictly speaking, this is the bosonic limit for the first root $q = \exp(2\pi i/F)$. In the context of a fractional superspace formalism^[3], we can choose to work with real θ

and ∂ , whereupon the consistency of the relation (2) under hermitian conjugation implies that α must be real. In the following sections, we will not be concerned with the $F \rightarrow \infty$ limit, so we let α remain unfixed.

The definition (2) implies

$$\partial \cdot \theta^n = \alpha(1 - q^n) \theta^{n-1} + q^n \theta^n \partial. \quad (5)$$

Setting $n = F$ in (5) demonstrates that the consistency of the formalism requires the condition (3). We shall also need the operator $B_{(F)}$ defined as

$$B_{(F)} = \sum_{i=0}^{\infty} c_i \theta^i \partial^i = c_0 + \sum_{i=1}^{F-1} c_i \theta^i \partial^i \quad (6a)$$

with

$$c_0 = (1 - F)/2 \quad \text{and} \quad c_i = [\alpha^i (1 - q^i)]^{-1} \quad (i = 1, 2, \dots, F-1). \quad (6b)$$

This operator has the following properties:

$$[B_{(F)}, \theta] = \theta, \quad [B_{(F)}, \partial] = -\partial. \quad (7)$$

One may also introduce other derivatives which satisfy q -deformed relations between themselves (see Ref. [4]).

A matrix realization of θ and ∂ is given by

$$\theta = \begin{pmatrix} 0 & a_1 & 0 & 0 & 0 \\ 0 & 0 & a_2 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & a_{F-1} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \partial = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ b_1 & 0 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & b_{F-1} & 0 \end{pmatrix} \quad (8a)$$

with the constraint (no summation on i)

$$a_i b_i = \alpha(1 - q^{-i}). \quad (8b)$$

Note that in general $\partial \neq \theta^\dagger$. In this matrix realization, $B_{(F)}$ is found to be the third component of the spin- $(F-1)/2$ representation of the rotational group:

$$B_{(F)} = J_3^{[(F-1)/2]}. \quad (9)$$

3. Fractional supersymmetric quantum mechanics

FSUSY quantum mechanics of order F is defined through the algebra

$$Q^F = H, \quad [H, Q] = 0, \quad F = 2, 3, \dots \quad (10)$$

where H is the Hamiltonian and Q the conserved fractional supercharge. The construction of Q and H will be realized in terms of paragrassmann variables (θ, ∂) of order F . Let us first work in one dimension. We introduce the bosonic operators a and a^\dagger

$$a = [p + iW(x)]/\sqrt{2}, \quad a^\dagger = [p - iW(x)]/\sqrt{2} \quad (11)$$

which satisfy

$$[a^\dagger, a] = \frac{d}{dx}W(x) \equiv W'(x) \quad (12)$$

where $p = -id/dx$. A Hamiltonian and a conserved charge satisfying (10) are given by:

$$Q = \partial^{F-1}a + eP\theta a^\dagger + (1-P)\theta \quad (13)$$

$$H = \frac{1}{2}(p^2 + W^2) + W' \cdot S \quad (14)$$

with

$$S = -\frac{1}{2} + P, \quad P = e\theta^{F-1}\partial^{F-1}, \quad P^2 = P \quad (15)$$

and where the order-dependent constant e is given by

$$e^{-1} = F\alpha^{F-1}. \quad (16)$$

In (13) and (15), it is understood that θ and ∂ satisfy the relations (1-3) for a given order F .

For $F = 2$, we have $P\theta = \theta$ and we recover for Q and for H (*i.e.*, for S) the ordinary supersymmetric quantities:

$$Q = \partial a + e\theta a^\dagger \quad (17)$$

and

$$S = -\frac{1}{2} + e\theta\partial. \quad (18)$$

With the realization (8), we find

$$S = \frac{1}{2}\sigma_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (19)$$

Thus the Hamiltonian (14) may be interpreted as describing a “spin-1/2” particle moving in a potential $W^2/2$ and a “magnetic field” W' . Moreover, with the choice $\alpha = 1/2$ ($e = 1$), we may choose $\theta = \sigma_+$ and $\partial = \sigma_-$, *i.e.*, $\partial = \theta^\dagger$.

In the general case, using the realization (8) in terms of $F \times F$ matrices, the projector P is found to be a matrix whose upper left entry is one and all others zero. Therefore, the matrix S is

$$S = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \quad (20)$$

Restricting ourself to the harmonic oscillator potential, *i.e.* $W = \pm\omega x$, the spectrum of the Hamiltonian (14) is clearly

$$E = (n + \frac{1}{2} \pm s)\omega, \quad n = 0, 1, 2, \dots \quad (21)$$

where s stands for the eigenvalues of S , which are: $s = +\frac{1}{2}$ (non-degenerate) and $s = -\frac{1}{2}$ ($F - 1$ degenerate). Unlike the SUSY case (and the PSUSY^[1,2] one; see below), the spectrum of the “spin” matrix is not invariant under $S \rightarrow -S$. Therefore the substitution $\omega \rightarrow -\omega$ leads to a different spectrum; hence the \pm sign in (21). Let us explicitly compare the spectra for the SUSY, PSUSY of order 3, and FSUSY of order 3 cases. The energy levels are given by the formula (21) with the following values for s :

$$\begin{aligned} \text{SUSY : } & s = \frac{1}{2}, -\frac{1}{2} \\ \text{PSUSY : } & s = 1, 0, -1 \\ \text{FSUSY : } & s = \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \end{aligned} \quad (22)$$

The spectra are shown in Fig. 1. The subscripts (+) and (−) refer to the two inequivalent possibilities $\pm s$ of the FSUSY construction [*c.f.* (21)]. In the SUSY and FSUSY_(−) contexts, there is a unique ground state with zero energy, with all

other levels respectively 2-fold and 3-fold degenerate. The only difference for the FSUSY₍₊₎ case is that the ground state is 2-fold degenerate. In the PSUSY case, the unique ground state has negative energy, the next level is 2-fold degenerate (and positive), and all others are 3-fold degenerate.

In general, we have the following for the FSUSY cases of arbitrary order. With the “minus” sign in (21), the spectrum is F -fold degenerate for all levels except for the *unique* ground state (which has zero energy). Thus, FSUSY is not spontaneously broken, *i.e.*, the ground state is invariant under FSUSY transformations since $Q|0\rangle = 0$. If this were not the case, one would have another state $|0\rangle' = Q|0\rangle$ with zero energy since Q commutes with H . On the other hand, with the “plus” sign, the ground state still has zero energy but is now $(F - 1)$ -fold degenerate, and FSUSY might be spontaneously broken. Like the SUSY case (but unlike the PSUSY case), the spectrum is non-negative. Unlike the SUSY case, however, it is possible to have a degenerate vacuum with zero energy. This is forbidden in SUSY since $H|0\rangle = 0$ implies $Q|0\rangle = 0$. Indeed, $0 = \langle 0|H|0\rangle = \langle 0|Q^2|0\rangle = |Q|0\rangle|^2$ since Q is hermitian.

It is straightforward to extend our construction to purely magnetic two-dimensional systems, *i.e.*, to particles moving in the plane (x, y) under the influence of a magnetic field B_z directed along the z -axis. Working with the (gauge-independent) components $v_x = p_x - A_x$ and $v_y = p_y - A_y$ of the velocity (where A_x and A_y are the two components of the potential vector), we have $[v_x, v_y] = iB_z$. Therefore, to transpose the formulas (11-14) from one- to two-dimensional systems, it suffices to make the substitutions $p \rightarrow v_x$ and $W \rightarrow v_y$ which imply $W' \rightarrow -B_z$. The Hamiltonian is then

$$H = \frac{1}{2}(v_x^2 + v_y^2) - B_z \cdot S \quad (23)$$

which is a gauge-independent expression.

Let us now turn to the $1/x^2$ potential, *i.e.*, $W = \lambda/x$. In that particular case, we have also a dynamical conformal symmetry, that of dilation, whose generator is

$$D = -\frac{1}{2}xp + \frac{i}{2}P_{(F)} - \frac{i}{F}B_{(F)}, \quad (24)$$

with $P_{(F)}$ given in (15) and $B_{(F)}$ in (6). D satisfies the following commutation relations:

$$[D, Q] = -\frac{i}{F}Q, \quad [D, H] = -iH. \quad (25)$$

For $F = 2$, we have $B_{(2)} = -\frac{1}{2} + P_{(2)}$ and we recover the usual dilation operator $D = -\frac{1}{2}xp + \frac{i}{4}$. In ordinary SUSY quantum mechanics, there are also other dynamical symmetries, namely, the special conformal (K) and superconformal (S) ones. For $F > 2$, this is not the case[†] since their existence would imply an infinite conformal algebra. To see this, remember first that a closed subalgebra of the (Neveu-Schwarz sector of the) Super-Virasoro algebra is the $OSp(2, 1)$ superalgebra generated by the bosonic generators H, D, K (*i.e.*, L_1, L_0, L_{-1} in the usual notation of the Virasoro generators) and the fermionic ones Q, S (*i.e.*, $G_{1/2}, G_{-1/2}$). [The bosonic operators alone form an $SO(2, 1)$ algebra.] In opposition, the maximal finite closed subalgebra of the *fractional* super-Virasoro algebra^[5] (for $F > 2$) is generated by H, D and Q (*i.e.*, L_1, L_0 and $G_{1/F}$) only. As soon as one adjoins L_{-1} or $G_{-1+1/F}$, the entire infinite algebra follows. For more details, see Refs. [5].

4. Fractional supersymmetry transformations

In this section, we describe a way of constructing FSUSY transformations within a pseudo-classical Lagrangian context, and we provide a FSUSY invariant action for the free particle. The interacting case will be discussed elsewhere.

Let us first recall the ordinary SUSY case. We work in one dimension. The position of the particle is described by $x(t)$, whereas its internal space is described by a *real* fermionic variable $\psi(t)$ which satisfies $\psi^2 = 0$. The SUSY transformations are given by

$$\delta x = i\epsilon \psi \tag{26a}$$

$$\delta \psi = \epsilon \dot{x} \tag{26b}$$

where ϵ is a fermionic infinitesimal parameter. Since $\epsilon\psi = -\psi\epsilon$, the r.h.s of Eqs. (26) are real. Note that $\delta^2 x = i\epsilon_1\epsilon_2 \dot{x}$ and $\delta^2 \psi = i\epsilon_1\epsilon_2 \dot{\psi}$, so SUSY transformations are indeed the square roots of time translations. An action invariant under (26) is

$$S = \int dt \frac{1}{2}(\dot{x}^2 + i\dot{\psi}\psi). \tag{27}$$

[†] Actually, $D' = -\frac{1}{2}xp + \frac{i}{4}$ and $K' = \frac{1}{2}x^2$ form an $SO(2, 1)$ algebra with H for any order F , but they have the wrong commutation relations with Q (for $F > 2$) and therefore do not generate proper subalgebras of the fractional Super-Virasoro algebra.

More precisely, the Lagrangian within (27) varies by the total time derivative $d[\frac{i}{2}\dot{x}\psi]/dt$. Note that since $\psi\dot{\psi} = -\dot{\psi}\psi$, this action is real. The variables x and ψ can be viewed as the components of a superspace coordinate $Z(t, \theta) = x + \psi\theta$ where θ is a real grassmann variable, *i.e.*, satisfying $\theta^2 = 0$. We say that x and ψ respectively belong to the sector-0 and sector-1 of the theory (θ is also in sector-1 and sectors are defined modulo 2).

We now turn to the next order, *i.e.*, *cube* roots of time translations. We need a theory with three sectors. The bosonic variable $x(t)$ remains a sector-0 quantity, but now we must have *two* types of *real* internal-space variables, $\phi(t)$ and $\psi(t)$, which respectively belong to sector-1 and sector-2. These three fields can be viewed as the components of a fractional superspace^[3] coordinate of order 3: $Z(t, \theta) = x + \psi\theta + \phi\theta^2$, where θ is a real paragrassmann variable satisfying $\theta^3 = 0$ (and belonging to the sector-1). Sectors are defined modulo 3. We take the following commutation relations between the new fields:

$$\psi\dot{\phi} = q\dot{\phi}\psi, \quad \dot{\psi}\phi = q\phi\dot{\psi} \quad (28)$$

where q is a primitive cube root of unity, *i.e.*, satisfies (3) with $F = 3$. The FSUSY transformations of order 3 are given by

$$\delta x = i\epsilon\alpha(1 - q)\psi \quad (29a)$$

$$\delta\psi = i\epsilon\alpha(1 - q^2)\phi \quad (29b)$$

$$\delta\phi = \epsilon e \dot{x} \quad (29c)$$

where α is a free (bosonic) constant and $e^{-1} = 3\alpha^2$. Now, ϵ is a sector-1 infinitesimal parameter which must satisfy

$$\epsilon x = x\epsilon, \quad \epsilon\phi = q\phi\epsilon, \quad \epsilon\psi = q^2\psi\epsilon. \quad (30)$$

We easily see that $\delta^3 = -\epsilon_1\epsilon_2\epsilon_3 d/dt$ and so Eqs. (29) do represent cube roots of time translations. The reality of the r.h.s of (29) follows from (30). An action invariant under (29) is

$$S = \int dt \frac{1}{2} [\dot{x}^2 + i\beta(1 - q)\dot{\phi}\psi + i\beta(1 - q^2)\dot{\psi}\phi] \quad (31)$$

with $\beta = 3\alpha^3$. This Lagrangian is real, and varies by a total derivative under the transformations (29) [see below].

Let us now turn to the general case, *i.e.*, the F^{th} roots of time translations with $F = 1, 2, \dots$. We need F *real* fields $\psi_{(i)}(t)$ [$i = 0, 1, \dots, F-1$] which belong to the sector- $(F-i)$, with $\psi_{(0)} \equiv x(t)$. Sectors are defined modulo F . These fields can be viewed as the components of a fractional superspace^[3] coordinate of order F : $Z(t, \theta) = \sum_{i=0}^{F-1} \psi_{(i)} \theta^i = x(t) + \sum_{i=1}^{F-1} \psi_{(i)} \theta^i$, where θ is a real paragrassmann variable satisfying $\theta^F = 0$ (and belonging to sector-1). We introduce the commutation relations,

$$\psi_{(i)} \psi_{(F-i)} = q^i \psi_{(F-i)} \psi_{(i)}, \quad (32)$$

where q is a primitive F^{th} root of unity, *i.e.*, satisfies (3). Taking the time-derivative of both sides of (32) and using the symmetry under $i \rightarrow F-i$, we get

$$\psi_{(i)} \dot{\psi}_{(F-i)} = q^i \dot{\psi}_{(F-i)} \psi_{(i)}. \quad (33)$$

Although (32) might be considered more fundamental, only (33) is really needed here (to ensure the reality of the action given below). No other commutation relations, *i.e.* between $\psi_{(i)}$ and $\psi_{(j)}$ with $j \neq F-i$, are required. For $F=2$ the relations (32) and (33) reduce [with the notation $\psi \equiv \psi_{(1)}$] to the proper SUSY results $\psi^2 = 0$ and $\psi \dot{\psi} = -\dot{\psi} \psi$. Note also that $\psi_{(0)}$ is indeed a commuting variable.

The FSUSY transformations of order F are given by ($i = 1, 2, \dots, F-1$):

$$\delta \psi_{(i-1)} = i\epsilon \alpha (1 - q^i) \psi_{(i)} \quad (34a)$$

$$\delta \psi_{(F-1)} = \epsilon e \dot{x} \quad (34b)$$

where $e^{-1} = F\alpha^{F-1}$ and where the infinitesimal parameter ϵ belongs as before to the sector-1. We now have $\delta^F \psi_{(i)} = i^{F-1} \epsilon_1 \dots \epsilon_F \dot{\psi}_{(i)}$, since $\prod_{i=1}^{F-1} (1 - q^i) = F$. Our previous results (26) and (29) are recovered for $F=2$ and $F=3$ (with the notation $\psi_{(1)} \equiv \psi$ and $\psi_{(2)} \equiv \phi$). For $F=1$, we simply have $\delta x = \epsilon \dot{x}$. To ensure that the r.h.s of the transformations (34) are real (and that the action given below is invariant), we must take the following commutation relations between ϵ and $\psi_{(i)}$:

$$\epsilon \psi_{(i)} = q^{-i} \psi_{(i)} \epsilon. \quad (35)$$

An action invariant under (34) is:

$$S = \int dt \frac{1}{2} \left[\dot{x}^2 + i\beta \sum_{i=0}^{F-1} (1 - q^{-i}) \dot{\psi}_{(i)} \psi_{(F-i)} \right] \quad (36a)$$

with $\beta = F\alpha^F$. More precisely, the Lagrangian varies by the total derivative

$$\delta L = \frac{d}{dt} \frac{\epsilon}{2} [i\alpha(1-q) \dot{x} \psi_{(1)} + \dot{x}^2 \delta_{F,1}]. \quad (37)$$

Alternatively, using (33) and the symmetry of the action under the substitution $i \rightarrow F - i$, we may rewrite (36a) as

$$S = \int dt \frac{1}{2} \left[\dot{x}^2 - i\beta \sum_{i=0}^{F-1} (1 - q^{-i}) \psi_{(i)} \dot{\psi}_{(F-i)} \right]. \quad (36b)$$

Note that the action is real. The first term of the sum in (36) always vanishes, but is included because it allows us to take the $F = 1$ limit, which corresponds to the spinless particle: $S = \int dt \frac{1}{2} \dot{x}^2$. The cases $F = 2$ (with $\alpha = 1/2$) and $F = 3$ reduce to those given previously in (27) and (31).

One may go further and construct the conserved Nöther charges associated with these FSUSY transformations.^[3,7] In Ref. [3], we also give a fractional-superspace formulation of the action (36), in order to make its FSUSY invariance manifest.

5. Conclusion

We conclude with a general remark. The SUSY algebra $Q^2 = H$, the PSUSY one $Q^3 = QH$ and the FSUSY one $Q^3 = H$ are concrete realizations of subalgebras of respectively the superVirasoro, the para-superVirasoro and fractional superVirasoro algebras^[4]. Recently, these generalizations of the superVirasoro algebra have been incorporated within a unified formalism, written in terms of fractional superspace coordinates, under the name *generalized super-Virasoro algebras*^[4]. This construction also contains new types of algebras such as the following paragrassmann extention of the FSUSY algebra of order 3: $Q^4 = QH$ with $Q^3 \neq 0$. Quantum-mechanical realizations of such algebras will be discussed elsewhere.

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